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NOTE

RECONSTRUCTIBILITY VERSUS EDGE RECONSTRUCTIBILITY OF INFINITE GRAPHS

Carsten THOMASSEN
Matematisk Institut, Universitetsparken, 8000 Aarhus C, Denmark

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For every cardinal $\alpha > \aleph_0$ there exists an α-regular graph which is reconstructible but not edge reconstructible.

Greenwell [2] proved that, for a finite graph G without isolated vertices, reconstructibility of G implies edge reconstructibility of G. Bondy and Hemminger [1] obtained this from a general theorem on reconstruction of finite graphs and asked if any infinite graph G with no isolated vertices is edge reconstructible provided it is reconstructible. All classes of infinite graphs described in [1] and, in addition, the graphs considered in [4] which are known to be reconstructible can also be shown to be edge reconstructible. Furthermore, all counterexamples to the Edge Reconstruction Conjecture (for infinite graphs) described in [3] are also counterexamples to the Reconstruction Conjecture. However, by modifying these graphs we answer the question of Bondy and Hemminger in the negative.

Theorem. *For each pair α, β of cardinals such that $\alpha > \beta \geq \aleph_0$, there exists a graph $G^{\alpha,\beta}$ which is regular of degree α and has β components (and hence α vertices and α edges) such that $G^{\alpha,\beta}$ is reconstructible but not edge reconstructible.*

Proof. The proof is based on properties of the graphs $H(\alpha, k)$ and $G(\alpha, k)$ of [3] with which familiarity is assumed. We define $L(\alpha, \beta, k)$ as the graph obtained from $G(\alpha, k)$ by first deleting the set $D(\alpha, k)$ of isolated vertices of $G(\alpha, k)$ (the resulting graph is denoted $G'(\alpha, k)$) and then adding, for each endvertex x of $B(\alpha, k)$, a T_α (the α-regular tree) and identifying one of its vertices with x. Finally we add βT_α, i.e. β copies of T_α. Thus $L(\alpha, \beta, k)$ is regular of degree α and it has β components. We may regard $H(\alpha, k)$ as the subgraph of $L(\alpha, \beta, k)$ induced by those vertices of $L(\alpha, \beta, k)$ which are contained in triangles of $L(\alpha, \beta, k)$ and which are equivalent (in the sense of [3]) to only finitely many vertices of $L(\alpha, \beta, k)$. Thus an isomorphism

$$\sigma : L(\alpha, \beta, k) \to L(\alpha, \beta, m)$$

induces an isomorphism

$$\sigma': H(\alpha, k) \to H(\alpha, m)$$

which implies, by [3], that $k = m$. Also it follows from the discussion of [3] that for every edge e of $L(\alpha, \beta, k)$

$$L(\alpha, \beta, k) - e \simeq L(\alpha, \beta, m) \quad \text{for some} \quad m \geq k - 2,$$

and that for each $m \geq \max(k-1, 0)$

$$L(\alpha, \beta, k) - e \simeq L(\alpha, \beta, m)$$

for α edges e. Thus $L(\alpha, \beta, 1)$ and $L(\alpha, \beta, 0)$ have the same families of edge-deleted subgraphs but are non-isomorphic. In particular, $G^{\alpha,\beta} = L(\alpha, \beta, 0)$ is not edge reconstructible.

In order to complete the proof we prove that $G^{\alpha,\beta}$ is reconstructible. From the discussion of [3] it follows that for any vertex x in $G'(\alpha, k) - B(\alpha, k)$,

$$L(\alpha, \beta, k) - x \simeq L(\alpha, \beta, m) \quad \text{for some} \quad m \geq k - 1,$$

and that for any x in $B(\alpha, k)$,

$$L(\alpha, \beta, k) - x \simeq \alpha T_\alpha \cup L(\alpha, \beta, m) \quad \text{for some} \quad m \geq k.$$

Clearly, if x is a vertex of $L(\alpha, \beta, k)$ not in $G'(\alpha, k)$, then

$$L(\alpha, \beta, k) - x \simeq \alpha T_\alpha \cup L(\alpha, \beta, k).$$

Now suppose H is a graph and $\sigma: V(G^{\alpha,\beta}) \to V(H)$ is a bijection such that $G^{\alpha,\beta} - x \simeq H - \sigma(x)$ for every vertex x of $G^{\alpha,\beta}$. Consider a vertex x of $G^{\alpha,\beta}$ such that $G^{\alpha,\beta} - x \simeq G^{\alpha,\beta}$. Then

$$G^{\alpha,\beta} \simeq G^{\alpha,\beta} - x \simeq H - \sigma(x).$$

In particular, $H - \sigma(x)$ contains an infinite set of mutually equivalent vertices (namely one of the sets C_F defined in [3]) and hence H also contains such a set. For any vertex y in such a set, we have

$$H \simeq H - y \simeq G^{\alpha,\beta} - \sigma^{-1}(y) \simeq L(\alpha, \beta, k) \quad \text{for some} \quad k.$$

We regard H as being equal to $L(\alpha, \beta, k)$. Consider a vertex x of $G^{\alpha,\beta} = L(\alpha, \beta, 0)$ which is not in $G'(\alpha, 0)$. Since $G^{\alpha,\beta} - x$ has α components it follows that $\sigma(x)$ is not in $G'(\alpha, k) - B(\alpha, k)$. By a remark above,

$$L(\alpha, \beta, k) - \sigma(x) \simeq L(\alpha, \beta, m) \cup \alpha T_\alpha \quad \text{for some} \quad m \geq k.$$

But also

$$L(\alpha, \beta, k) - \sigma(x) \simeq L(\alpha, \beta, 0) \cup \alpha T_\alpha$$

so $m = k = 0$, i.e., $H \simeq L(\alpha, \beta, 0) = G^{\alpha,\beta}$ and hence $G^{\alpha,\beta}$ is reconstructible.

We do not know if there are countable graphs which are reconstructible but not edge reconstructible and we do not know if there are connected graphs with these properties.

We do not even know if the Edge Reconstruction Conjecture holds for infinite connected graphs but we believe that it does not.

References

[1] J.A. Bondy and R.L. Hemminger, Graph reconstruction—a survey, J. Graph Theory 1 (1977) 227–268.

[2] D.L. Greenwell, Reconstructing graphs, Proc. Amer. Math. Soc. 30 (1971) 431–433.

[3] C. Thomassen, Counterexamples to the edge reconstruction conjecture for infinite graphs, Discrete Math. 19 (1977) 293–295.

[4] C. Thomassen, Reconstructing 1-coherent locally finite trees, Comment. Math. Helv. (to appear).